

Dark Solitons, D-branes and Noncommutative Tachyon Field Theory

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ABSTRACT: In this paper we discuss the boson/vortex duality by mapping the Gross-Pitaevskii theory into an effective string theory, both with and without boundaries. Through the effective string theory, we find the Seiberg-Witten map between the commutative and the noncommutative tachyon field theories, and consequently identify their soliton solutions with the D-branes in the effective string theory. We perform various checks of the duality map and the identification of classical solutions. This new insight of the duality between the Gross-Pitaevskii theory and the effective string theory allows us to test many results of string theory in Bose-Einstein condensates, and at the same time help us understand the quantum behavior of superfluids and cold atom systems.

KEYWORDS: Bose-Einstein Condensates, Effective String Theory, Dark Solitons, D-branes, Duality, Noncommutative Tachyon Fields, Noncommutative Soliton

Contents

1	Introduction	1
2	Commutative Field Theory	4
2.1	Abelian Higgs Model	4
2.2	Commutative Tachyon Field Theory	5
2.2.1	Dark Soliton	6
2.2.2	Bright Soliton	8
2.2.3	Vortex Line	8
2.2.4	Vortex Ring	9
2.3	Some Remarks	9
3	Boson-Vortex Duality	12
3.1	Duality without Boundary	12
3.2	Duality with Boundary	15
3.3	Generalizations	20
4	Solitons and D-branes	22
4.1	Noncommutative Tachyon Field Theory	23
4.2	Identification of Classical Solutions	25
4.2.1	D-brane Tension	26
4.2.2	D-brane Interaction	27
5	Discussions	30
A	Some Details in the Duality Map	31

1 Introduction

It was known long ago that there are some excitations in Bose-Einstein condensates (BEC), such as vortex lines, vortex rings and dark solitons, which are very similar to some basic ingredients such as open strings, closed strings and D-branes in string theory.

In Refs. [1, 2], the comparison between the dark solitons and the D-branes was made quantitatively. The authors studied a two-component BEC model, and found the energy of this system is the same as a 4-dimensional $\mathcal{N} = 2$ supersymmetric sigma model [3]. Using the BPS procedure, they found the soliton solution and the vortex solution of the system. However, in the configuration they found the soliton solution is not the standard dark soliton solution known in the Bose-Einstein condensates, instead it is the boundary between two components of the BEC.

More recently, some numerical work in BEC [4] demonstrated that the vortex lines can attach directly to dark solitons, which mimics the configuration of open strings attached to D-branes in string theory. Hence, it is very conceivable that there should be some explanations about this similarity from theoretical point of view. If this correspondence can be put on solid ground, one may expect to simulate string theory in a BEC system, and at the same time bring in some new ideas to the study of Bose-Einstein condensates.

We would like to explore this relation between BEC and string theory from theoretical perspective in a series of papers. Our starting point is following. The Gross-Pitaevskii equation is known as an effective theory to describe Bose-Einstein condensates [5], and it has been shown [6–8] that for a spacetime without boundary the Gross-Pitaevskii equation can be mapped into an effective theory, which is very similar to the standard string theory in the large B -field limit. In this paper, we will demonstrate that this duality can also be generalized to a spacetime with boundary, where certain soliton solutions play the role of the boundary. We would like to analyze this effective string theory from different perspectives. As a first step, we study the soliton solution in the Gross-Pitaevskii theory and the D-brane solution in the effective string theory. Based on the duality of the Gross-Pitaevskii theory and the effective string theory, we provide a proof of identifying dark solitons with D-branes.

To obtain the soliton solutions in different theories, it is convenient to apply the technique of noncommutative geometry. This approach was first introduced in string theory by Seiberg and Witten [9]. They studied the Yang-Mills theory from the open string sector in the large B -field limit. They found that there are both commutative and noncommutative descriptions of the Yang-Mills theory, and the gauge fields in two descriptions are related by a nonlinear map (Seiberg-Witten map).

A similar relation should be obtained for the tachyon field. If we think of the Gross-Pitaevskii theory as the commutative description of the tachyon field, from the discussions above, we have seen that one can map it into an effective string theory in the large B -field limit. Hence, the Gross-Pitaevskii theory can be viewed as the tachyon field theory derived from a string theory. Next, we can follow the approach of Refs. [10–14] to derive a noncommutative tachyon field theory from the effective string theory. The commutative and the noncommutative tachyon fields can be related by a Seiberg-Witten map. We demonstrate the relations in Fig. 1.

To find the Seiberg-Witten map for the tachyon field, it is more convenient to first gauge the scalar field theory, and study the commutative and the noncommutative Abelian Higgs models. The commutative Abelian Higgs model has been studied in the literature in great detail. It is known that it has some topologically nontrivial solutions such as the Nielsen-Olesen vortex line [15], which becomes the vortex line solution in the Gross-Pitaevskii theory when the gauge field is turned off, and the endpoints of the Nielsen-Olesen string have to terminate at (anti-)monopoles [16].

On the other hand, Ref. [17] has studied the (2+1)-dimensional noncommutative Abelian Higgs model and found its vortex solutions. It is straightforward to generalize the analysis to our case, and find the Seiberg-Witten map between the scalars in the commutative and the noncommutative Abelian Higgs models. We can then turn off the gauge

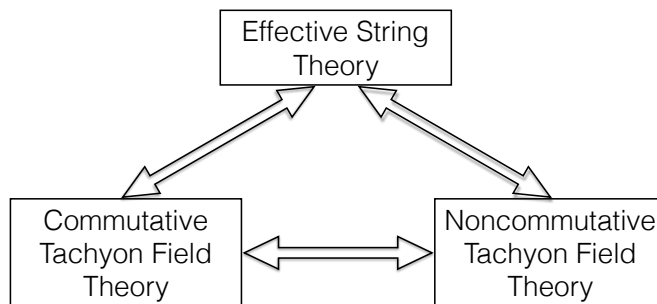


Figure 1. The relation between different theories

fields to obtain the Seiberg-Witten map for pure scalar field theories. We will see that the map becomes a trivial linear map. The theories with commutative and noncommutative scalars are just the commutative and the noncommutative tachyon field theories that we are looking for.

The next problem is to find the soliton solution in each description and identify them. For the commutative tachyon field theory, or the Gross-Pitaevskii theory, the dark soliton solution is well-known. A more field-theoretic approach is to apply the BPS procedure to find the lowest energy configurations that connect different vacua at different boundaries. This approach allows one to transform the original second-order nonlinear differential equation into a first-order linear differential equation. For the noncommutative tachyon field theory, Ref. [18] has studied the noncommutative soliton solutions, and furthermore, Ref. [19] has shown that the soliton solutions in the noncommutative tachyon field theory can be identified with the D-brane solutions in string theory.

Taking all the descriptions into account, we not only can identify the D-branes in string theory and the noncommutative solitons in the noncommutative tachyon field theory, but we may also identify them with the soliton solutions in the commutative tachyon field theory, i.e. the dark soliton solutions in the Gross-Pitaevskii theory. Therefore, we achieve the goal of identifying the D-branes in string theory and the dark solitons in Bose-Einstein condensates.

This paper is organized as follows. In Section 2, we review the commutative field theory. First, the commutative Abelian Higgs model and the Nielsen-Olesen vortex line solution will be discussed in Subsection 2.1. By turning off the gauge field, we obtain the commutative tachyon field theory, i.e. the Gross-Pitaevskii theory. We will review various solutions of this theory in Subsection 2.2. Some issues like the stability and the generalization to higher dimensions will be discussed in Subsection 2.3. In Section 3, we map the Gross-Pitaevskii theory into an effective string theory. The duality for the spacetime without boundary will be reviewed in Subsection 3.1, and we will generalize this duality to the spacetime with boundary in Subsection 3.2. To close the triangle relation shown in Fig. 1, we discuss the noncommutative tachyon field theory and its soliton solution in Subsection 4.1. Following the logic discussed above, we can identify the dark soliton,

the noncommutative soliton and the D-brane with each other. To test this identification, we make various checks in Subsection 4.2, including the D-brane tension from the three descriptions and the comparison of the D-brane interaction with the numerical results of the dark soliton interaction. Finally, we discuss the future directions in Section 5.

2 Commutative Field Theory

In this section, we would like to review some known results of the scalar field theory. Technically, it would be easy to discuss the gauged scalar field theory, hence we first discuss the ordinary commutative Abelian Higgs model and the Nielsen-Olsen vortex line solution in Subsection 2.1. By turning off the gauge field, we obtain the commutative tachyon field theory, i.e. the Gross-Pitaevskii theory. We will review this theory and some classical solutions in Subsection 2.2. In Subsection 2.3 we discuss some issues such as the stability and the generalization to higher dimensions.

2.1 Abelian Higgs Model

The ordinary commutative Abelian Higgs model in (3+1)-dimensions is given by the Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}|(\partial_\mu + ieA_\mu)\phi|^2 - \lambda(|\phi|^2 - |\phi_0|^2)^2, \quad (2.1)$$

where ϕ is a complex scalar, and A_μ is the gauge field. The equations of motion are

$$(\partial_\mu + ieA_\mu)^2\phi + 4\lambda(|\phi|^2 - |\phi_0|^2)\phi = 0, \quad (2.2)$$

$$\partial^\nu F_{\mu\nu} = \frac{i}{2}e(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*) + e^2A_\mu\phi^*\phi. \quad (2.3)$$

This theory has nontrivial classical solutions, including string-like Nielsen-Olsen vortex lines. The Nielsen-Olsen vortex line solution (or Nielsen-Olsen string) was found first by Nielsen and Olsen in Ref. [15]. Under the gauge $A_0 = 0$, considering a string-like solution, i.e. preserving a cylindrical symmetry, one can assume the axis along the z -direction and apply the ansatz

$$\mathbf{A}(\mathbf{r}) = \frac{\mathbf{r} \times \mathbf{e}_z}{r} |\mathbf{A}(r)|, \quad (2.4)$$

and the flux carried by the Nielsen-Olsen vortex line is

$$\Phi(r) = \oint A_\mu(x) dx^\mu = 2\pi r |\mathbf{A}(r)|, \quad (2.5)$$

Since the covariant derivative on the scalar field vanishes outside the vortex line, the phase of ϕ defined by $\phi = |\phi| e^{i\eta}$ satisfies

$$d\eta + eA = 0 \quad \text{for } r \rightarrow \infty. \quad (2.6)$$

Hence, the property that ϕ should be single-valued requires

$$\Phi = \lim_{r_0 \rightarrow \infty} \oint_{r=r_0} A = - \lim_{r_0 \rightarrow \infty} \frac{1}{e} \oint_{r=r_0} d\eta = n \frac{2\pi}{e}, \quad (2.7)$$

i.e. the magnetic flux carried by the Nielsen-Olsen vortex line is quantized.

By plugging the ansatz (2.4) into the Lagrangian (2.1) and performing the variation of the fields, one can obtain the classical equations of motion for the configurations with cylindrical symmetry, which can be solved numerically. The solutions have the asymptotic behavior:

$$|\phi| = |\phi_0| = \text{const}, \quad |\mathbf{A}| = \frac{1}{er} + \frac{c}{e} K_1(er|\phi|), \quad \text{for } r \rightarrow \infty; \quad (2.8)$$

$$|\phi| = 0, \quad \text{for } r \rightarrow 0. \quad (2.9)$$

As discussed in Ref. [16], an infinitely long vortex line has infinite energy, hence unphysical. In order to have a finite length, a vortex line can terminate at a magnetic (anti-)monopole, which has the magnetic charge $g = n2\pi/e$. Hence, the magnetic flux carried by the Nielsen-Olsen vortex line can be absorbed by the monopole anti-monopole pair. Moreover, one can demonstrate that in this case the potential between the monopole anti-monopole pair is linear in the distance between them, i.e., the Nielsen-Olsen vortex line realizes the confinement in the Abelian Higgs model. Schematically, the configuration of a finite Nielsen-Olsen vortex line with a monopole anti-monopole pair at two endpoints is shown in Fig. 2.

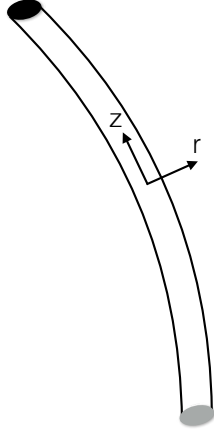


Figure 2. The sketch of the Nielsen-Olsen vortex line ending on a monopole anti-monopole pair

It is also known that the Nielsen-Olsen vortex line can be effectively described by the Nambu string action [15, 16], which also confirms our following discussions that the scalar theory can be mapped into an effective string theory. As we will also see later, the vortex line solution to the Gross-Pitaevskii solution can be viewed as the Nielsen-Olsen vortex line solution when the gauge field is turned off.

2.2 Commutative Tachyon Field Theory

If we turn off the gauge field, the commutative Abelian Higgs model (2.1) becomes the corresponding commutative scalar field theory:

$$\mathcal{L}_c = -\frac{1}{2}|\partial_\mu\phi|^2 - \lambda(|\phi|^2 - |\phi_0|^2)^2, \quad (2.10)$$

where the subscript “c” stands for the commutative theory, in contrast to the noncommutative theory that we will discuss later.

If we consider a non-relativistic theory, the commutative scalar theory above can be viewed as the relativistic version of the Gross-Pitaevskii theory, which is given by the following non-relativistic Lagrangian:

$$\mathcal{L}_{GP} = i\phi^\dagger \partial_t \phi - \frac{1}{2m} (\nabla \phi^\dagger)(\nabla \phi) - \frac{g}{2} (|\phi|^2 - \rho_0)^2. \quad (2.11)$$

Varying it with respect to ϕ^\dagger , we obtain a version of the Gross-Pitaevskii equation:

$$i \partial_t \phi + \frac{1}{2m} \nabla^2 \phi - g (|\phi|^2 - \rho_0) \phi = 0. \quad (2.12)$$

The Gross-Pitaevskii equation has various solutions with nontrivial topology [5]. Let us discuss them separately in the following.

2.2.1 Dark Soliton

The classical soliton solutions can be found by solving the Gross-Pitaevskii equation directly. A more field-theoretic way of finding the soliton solutions is to use the standard BPS approach, which we will briefly review now.

Based on the famous Derrick’s theorem, a stable soliton solution for the pure scalar theory exists only for dimensions $D \leq 2$. Hence, we restrict our discussions to the (1+1)D solutions in the following, i.e., we assume that the soliton solutions are independent of two spatial dimensions. The BPS procedure for the (1+1)D scalar field theory can be summarized as follows.

A general scalar field theory is given by

$$\mathcal{L} = -\frac{1}{2} (\partial_x \phi)^2 - V(\phi), \quad (2.13)$$

which leads to the field equation

$$\partial_x^2 \phi - V'(\phi) = 0. \quad (2.14)$$

If the potential $V(\phi)$ can be expressed as

$$V = (W')^2, \quad (2.15)$$

the energy of the system is given by

$$E = \int_{-\infty}^{\infty} dx \left[\frac{1}{2} (\partial_x \phi)^2 + (W')^2 \right], \quad (2.16)$$

where W is a functional of the field ϕ , and

$$W' \equiv \frac{\partial W}{\partial \phi}. \quad (2.17)$$

Consequently,

$$\begin{aligned}
E &= \int_{-\infty}^{\infty} dx \left[\left(\frac{1}{\sqrt{2}} \partial_x \phi - W' \right)^2 + \sqrt{2} W' \partial_x \phi \right] \\
&= \int_{-\infty}^{\infty} dx \left[\left(\frac{1}{\sqrt{2}} \partial_x \phi - W' \right)^2 + \sqrt{2} \frac{\partial W}{\partial x} \right] \\
&= \int_{-\infty}^{\infty} dx \left[\left(\frac{1}{\sqrt{2}} \partial_x \phi - W' \right)^2 \right] + \sqrt{2} [W(+\infty) - W(-\infty)] .
\end{aligned} \tag{2.18}$$

If $W(+\infty)$ and $W(-\infty)$ correspond to different vacua, the configuration provides a soliton with nontrivial topology, which is given by the solution of the first-order differential equation

$$\partial_x \phi = \sqrt{2} W' . \tag{2.19}$$

Eq. (2.19), which is also called the BPS equation, implies the field equation, since

$$\partial_x^2 \phi = \partial_x (\sqrt{2} W') = \sqrt{2} W'' \frac{\partial \phi}{\partial x} = \sqrt{2} W'' \sqrt{2} W' = 2W' W'' , \tag{2.20}$$

which is exactly the field equation (2.14):

$$\partial_x^2 \phi = V' = 2W' W'' . \tag{2.21}$$

Now let us come back to the discussion of the soliton solutions to the Gross-Pitaevskii equation. For the repulsive interaction, i.e. $g > 0$, the energy for the Gross-Pitaevskii equation is

$$E = \int_{-\infty}^{\infty} dx \left[\frac{\hbar^2}{2m} \left| \frac{d\Psi_0}{dx} \right|^2 + \frac{g}{2} (|\Psi_0|^2 - n)^2 \right] , \tag{2.22}$$

where

$$\Psi_0 = \sqrt{n} f \exp \left[-\frac{i\mu t}{\hbar} \right] \tag{2.23}$$

with the chemical potential μ , and f is in general a complex function

$$f = f_1 + i f_2 . \tag{2.24}$$

We choose $f_2 = \frac{v}{c}$, and define

$$\phi \equiv \frac{\hbar}{\sqrt{m}} \Psi_0 , \tag{2.25}$$

then the energy becomes

$$E = \int_{-\infty}^{\infty} dx \left[\frac{1}{2} \left| \frac{d\phi}{dx} \right|^2 + \frac{g}{2} \left(\frac{m}{\hbar^2} |\phi|^2 - n \right)^2 \right] . \tag{2.26}$$

Similar to what we discussed before, the BPS equation for the energy functional given by Eq. (2.26) can be written as follows:

$$\frac{d\phi}{dx} = \sqrt{g} \left(n - \frac{m}{\hbar^2} |\phi|^2 \right) \quad \text{or} \quad \frac{d\phi}{dx} = \sqrt{g} \left(\frac{m}{\hbar^2} |\phi|^2 - n \right) , \tag{2.27}$$

but the imaginary part of ϕ should be constant, in order that the energy functional has the expression of Eq. (2.18). The solutions to these two equations only differ by a minus sign. Let us consider the first equation, which is equivalent to

$$\begin{aligned} \frac{\hbar}{\sqrt{m}} \frac{df}{dx} &= \sqrt{gn}(1 - |f|^2) \\ \Rightarrow \frac{\hbar}{\sqrt{m}} \frac{df_1}{dx} &= \sqrt{gn}(1 - \frac{v^2}{c^2} - f_1^2), \quad \frac{\hbar}{\sqrt{m}} \frac{df_2}{dx} = 0. \end{aligned} \quad (2.28)$$

For $v = 0$ the equation above simplifies to

$$\sqrt{2\xi} \frac{df_1}{dx} = 1 - f_1^2, \quad (2.29)$$

where $\xi \equiv \hbar/\sqrt{2mgn}$ is the healing length. The solution to this equation is the dark soliton:

$$\Psi_0(x) = \sqrt{n} \tanh \left[\frac{x}{\sqrt{2\xi}} \right]. \quad (2.30)$$

If we perform a Galilean boost to the first one of Eqs. (2.28) using the method described in Ref. [20], it becomes

$$\sqrt{2\xi} \frac{df_1}{dx'} = 1 - \frac{v^2}{c^2} - f_1^2, \quad (2.31)$$

where $x' \equiv x - vt$. This new equation is exactly the same as Eq. (5.55) for an arbitrary constant v in Ref. [5], and the solution to this equation is

$$\Psi_0(x - vt) = \sqrt{n} \left(i \frac{v}{c} + \sqrt{1 - \frac{v^2}{c^2}} \tanh \left[\frac{x - vt}{\sqrt{2\xi}} \sqrt{1 - \frac{v^2}{c^2}} \right] \right), \quad (2.32)$$

which includes both the dark soliton solution ($v = 0$) and the grey soliton solution ($v \neq 0$).

2.2.2 Bright Soliton

When the interaction is attractive, i.e. $g < 0$, there is another kind of soliton solution to the Gross-Pitaevskii equation, which is called the bright soliton and has the form

$$\Psi(z) = \Psi(0) \frac{1}{\cosh(z/\sqrt{2\xi})}, \quad (2.33)$$

where $n_0 = |\Psi(0)|^2$ is the central density, and $\xi \equiv \hbar/\sqrt{2m|g|n_0}$.

2.2.3 Vortex Line

The Gross-Pitaevskii equation has another string-like solution called the vortex line. It can be viewed as the Nielsen-Olsen vortex line solution in the Abelian Higgs model in the limit of zero gauge field. In this subsection, we follow Ref. [5] to review this kind of solution.

To see the vortex line solution, we start with the Gross-Pitaevskii equation (2.12). Plugging the ansatz

$$\phi(\mathbf{r}, t) = \phi(\mathbf{r}) \exp \left(-\frac{i\mu t}{\hbar} \right) \quad (2.34)$$

in to Eq. (2.12), where μ is the chemical potential, we obtain

$$\left(-\frac{\hbar^2 \nabla^2}{2m} - \mu + g|\phi(\mathbf{r})|^2\right) \phi(\mathbf{r}) = 0. \quad (2.35)$$

For a string-like solution, we can introduce the cylindrical coordinates (r, φ, z) and further parametrize ϕ as

$$\phi = \sqrt{n} f(\eta) e^{is\varphi}, \quad (2.36)$$

where $\eta = r/\xi$ with $\xi \equiv \hbar/\sqrt{2m|g|n}$, and s is an integer characterizing the angular momenta carried by a vortex line. With this parametrization, one obtains the equation for $f(\eta)$:

$$\frac{1}{\eta} \frac{d}{d\eta} \left(\eta \frac{df}{d\eta} \right) + \left(1 - \frac{s^2}{\eta^2} \right) f - f^3 = 0, \quad (2.37)$$

and the boundary conditions are

$$\begin{aligned} f &\rightarrow 1, & \text{when } \eta &\rightarrow \infty; \\ f &\sim \eta^{|s|}, & \text{when } \eta &\rightarrow 0. \end{aligned} \quad (2.38)$$

The equation above can be solved numerically for a given value of s . Once the solution $f(\eta)$ is obtained, the energy of this configuration is

$$E = \frac{L\pi\hbar^2 n}{m} \int_0^{R/\xi} \eta d\eta \left[\left(\frac{df}{d\eta} \right)^2 + \frac{s^2}{\eta^2} f^2 + \frac{1}{2} (f^2 - 1)^2 \right], \quad (2.39)$$

where L and R are the effective length of the vortex line and the radius of the system respectively.

2.2.4 Vortex Ring

Similar to the vortex line solution discussed in the previous subsection, there is also the vortex ring solution, which does not have two endpoints, instead it is a closed string-like solution. In contrast to the vortex line solution, the vortex ring cannot be at rest. Moreover, as discussed in Ref. [5], the radius of the vortex ring can be much larger than the healing length ξ or comparable to the healing length ξ . Two parallel vortex rings with opposite circulation can also form a vortex pair, which was studied in Ref. [8] using the boson-vortex duality.

2.3 Some Remarks

In this section we would like to reconsider the solutions described above from their topological properties. This analysis is well-known, but it will allow us to discuss extended field configurations for D -dimensional Abelian Higgs model in full generality. For the time being we reintroduce the 1-form gauge field A , whose basic role is to remove the singularities in some of these solutions. Our assumption is that we can take A to be as weak as we wish at the expense of introducing an infrared cutoff in the theory.

We are going to consider purely static configurations in the gauge $A_0 = 0$, so there is no electric field. This means that the potential energy in Eq. (2.1) can be written in terms of the time-independent fields $\phi(\mathbf{x})$ and $A_i(\mathbf{x})$ as

$$E = \int d^{D-1}x \left[\frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} D_i \phi^\dagger D_i \phi + V(\phi \phi^\dagger) \right], \quad (2.40)$$

where

$$D_i \phi = \partial_\mu \phi + ie A_i \phi, \\ V(\phi \phi^\dagger) = \lambda (\phi \phi^\dagger - |\phi_0|^2)^2.$$

The expression of Eq. (2.40), in the limit when the electromagnetic potential is switched off, just reproduces the energies of particular configurations that were previously studied in context of the non-relativistic Gross-Pitaevskii model. We consider solutions which depend on $d \leq D - 1$ dimension. The remaining $D - d - 1$ dimensions just bring about an overall constant factor in Eq. (2.40), which may be regularized by putting the theory in a box and does not affect the classification from the point of view of topology.

The integrand in Eq. (2.40) is the sum of positive definite terms, each of which must give a finite contribution to the energy. It is implied that the potential $V(\phi \phi^\dagger)$ must vanish as $r \rightarrow \infty$, in particular, $|\phi| \rightarrow |\phi_0|$. The well-known Derrick's theorem states that stable topologically nontrivial nonsingular field configurations for a purely scalar field theory (i.e. $A_i = 0$ in our discussion) with second derivatives can exist only for $d < 2$.

$d = 1$ is the special case in which infinity is a disconnected domain with the topology of $\mathbb{S}_0 = \{+1, -1\}$. For the real scalar field theory (2.13), the set of values taken by ϕ at $+\infty$ can be obtained from the ones at $-\infty$ (and vice versa) by all the transformations in the \mathbb{Z}_2 group of reflections. This means that, due to the choice of boundary conditions we have a spontaneous breaking of \mathbb{Z}_2 , and topologically distinct configurations are classified by $\pi_0(\mathbb{Z}_2)$. The topologically nontrivial solution associated with this pattern of symmetry breaking (usually called domain wall) is just a solution to Eq. (2.19). Due to the fact that the broken group is discrete, there are no Goldstone modes associated with this symmetry breaking, and the description of domain walls using a theory of continuous fields is not expected. The dark soliton found in Subsection 2.2.1 for the complex scalar theory is unstable due to the fluctuations of the phase of ϕ . The BPS equation fixes ϕ up to an arbitrary constant phase, so that the vacuum manifold is continuous. This analysis is consistent with the fact that the homotopy group $\pi_0(U(1))$ is trivial.

For $d \geq 2$, in order to elucidate the obstruction brought about by the Derrick's theorem, we can switch on a gauge field. As previously stated, this requirement is not so restrictive, since we can always consider a weak gauge potential as we wish, so that the ungauged scalar theory with its solutions is a consistent physical limit of the gauged one. Infinity is now a connected topological domain S_{d-1} and a field configuration ϕ satisfying $V(\phi \phi^\dagger) = 0$ at infinity can be written as $\phi(\hat{\mathbf{r}}) = \gamma(\hat{\mathbf{r}}) \phi(\hat{\mathbf{r}}_1)$, where $\gamma(\hat{\mathbf{r}})$ is a transformation in $G = U(1)$ and $\phi(\hat{\mathbf{r}}_1)$ is the value of the field at some point at infinity, which is left unchanged by some subgroup H . We thus get a consistent description of ϕ as a mapping $S_{d-1} \mapsto G/H$ with

G/H denoting the coset space, and topologically distinct field configurations are classified by the homotopy group $\pi_{d-1}(G/H)$.

By studying the scaling of the terms in Eq. (2.40) in a way analog to the Derrick's theorem, one can show that for $0 < d < 4$ the energy given by Eq. (2.40) can have a minimum for a topologically nontrivial nonsingular field configuration.

Topological nontriviality is therefore related to both the dimension d and the coset group G/H . For $d = 2$, we see that $\pi_1(G/H)$ is nontrivial when G is a non-simply connected group like $U(1)$ or $SO(3)$, broken either completely or partially to a discrete subgroup. These topologically nontrivial configurations are usually identified with the cross-sections of vortex lines discussed in Subsections 2.1 and 2.2.3. This analysis shows that a vortex is defined by the presence of topologically nontrivial mappings of the circle S_1 into some coset space. By definition these mappings are the Goldstone bosons of G/H on the large cylinder S_1 surrounding the vortex, and they are twisted in such a way that they cannot be smoothly deformed into constants. If we try to reduce the radius of the sphere to zero, these Goldstone bosons must at some point become singular, or more precisely, at some point the parametrization of the field in terms of Goldstone bosons must break down. The singularity is actually not a physical one, because it occurs in a core (a line or perhaps a tube) within which the group G is no longer broken, so that the system is no longer described by Goldstone boson fields. This actually corresponds to the vanishing of the Higgs scalar field. In fact, the boundary conditions (2.38) have been chosen, so that $V(\phi\phi^\dagger) \rightarrow 0$ for $r \rightarrow \infty$ and the singularity of $\exp(is\varphi)$ with integer s at $r = 0$ is removed. The ansatz (2.36) is the most general one consistent with the cylindrical symmetry. There are two ways to interpret this ansatz. In the first one, φ is a scalar field taking values in $\mathbb{R}/2\pi s^{-1}\mathbb{Z}$, and there are no gauge transformations in this case. Alternatively, we can view φ as a function, whose values in $\mathbb{R}/2\pi\mathbb{Z}$ are defined modulo $2\pi s^{-1}\mathbb{Z}_s$, so that gauge transformations are actually shifts by values in $2\pi s^{-1}\mathbb{Z}_s$, i.e., they are \mathbb{Z}_s gauge transformations. So φ is actually a collection of functions $f_i : U_i \rightarrow \mathbb{R}/2\pi\mathbb{Z}$, such that on $U_{ij} = U_i \cap U_j$ we have $f_i - f_j = 2\pi s^{-1}m_{ij} \in 2\pi s^{-1}\mathbb{Z}_s$ and on $U_{ijk} = U_i \cap U_j \cap U_k$ the cocycle condition $m_{ij} + m_{jk} + m_{ki} = 0$ is satisfied. The exterior derivative $d\varphi$ is an exact one-form whose period is quantized

$$s \int d\varphi = 2\pi m \in 2\pi\mathbb{Z}_s. \quad (2.41)$$

There is still another description, whereby the field φ takes values in $\mathbb{R}/2\pi s^{-1}\mathbb{Z}$ and has the gauge transformation

$$\varphi \mapsto \varphi - \frac{e}{s}\lambda, \quad (2.42)$$

where λ is a $U(1)$ gauge parameter taking values in $\mathbb{S}^1 = \mathbb{R}/2\pi e^{-1}\mathbb{Z}$. To get an invariant action we have to switch on the gauge field A with a gauge transformation

$$A \mapsto A + d\lambda. \quad (2.43)$$

If φ acquires a vacuum expectation value, the $U(1)$ gauge group is broken down to \mathbb{Z}_s . Therefore, the phase factor $\exp(is\varphi)$ with $s > 1$ an integer number realizes a topologically

nontrivial mapping of \mathbb{S}^1 to the coset space $U(1)/\mathbb{Z}_s$. As a consequence, the magnetic flux through a contour encompassing the z -axis is quantized, as shown in Eq. (2.6) and Eq. (2.7). In fact, the symmetry breaking forces A to take the expectation value

$$A = -\frac{s}{e}d\varphi, \quad (2.44)$$

so that no continuous degrees of freedom survive, and we are left only with the discrete ones associated with \mathbb{Z}_s gauge symmetry.

Finally, let us consider the case $d = 3$, which is allowed by the scaling argument of the Derrick's theorem, but only under the assumption that the nontrivial topology may be attained. The $d = 3$ topologically nontrivial nonsingular configurations, realizing maps on the sphere $S_2 \mapsto G/H$, are classified by the homotopy group $\pi_2(G/H)$, which is nontrivial for a simply connected group G , such as $SU(2)$ broken to a subgroup H containing the $U(1)$ of electromagnetism. This is the case for magnetic monopoles in two-component scalar theories, however, in our case $G = U(1)$, the phase of the field φ on the sphere at infinity cannot make a jump when we go around a distant closed curve Γ , unless the phase has a discontinuity, which is not the case for smooth configurations. So up to a gauge transformation the field φ is asymptotically real, which means that the corresponding homotopy group is trivial and the given field can be pressed down to the vacuum.

3 Boson-Vortex Duality

In this section, we first briefly review in Subsection 3.1 the mapping of the Gross-Pitaevskii equation into an effective string theory in the spacetime without boundary. After that, in Subsection 3.2 we generalize the duality to the spacetime with boundary.

3.1 Duality without Boundary

To review the duality map, we follow closely Refs. [6–8]. Let us recall the Gross-Pitaevskii Lagrangian (2.11):

$$\mathcal{L}_{GP} = i\phi^\dagger \partial_t \phi - \frac{1}{2m}(\nabla \phi^\dagger)(\nabla \phi) - \frac{g}{2}(|\phi|^2 - \rho_0)^2.$$

We parametrize ϕ as

$$\phi = \sqrt{\rho} e^{i\eta}, \quad (3.1)$$

where η is the Goldstone boson, and ρ can be thought of as the Higgs boson. From this parametrization it is easy to derive

$$\begin{aligned} \phi^\dagger &= \sqrt{\rho} e^{-i\eta}, \\ \partial_t \phi &= \frac{\dot{\rho}}{2\sqrt{\rho}} e^{i\eta} + \sqrt{\rho} i e^{i\eta} \dot{\eta}, \\ \nabla \phi &= \frac{1}{2\sqrt{\rho}} e^{i\eta} (\nabla \rho) + \sqrt{\rho} i e^{i\eta} (\nabla \eta). \end{aligned} \quad (3.2)$$

Hence, the original Gross-Pitaevskii Lagrangian (2.11) becomes

$$\mathcal{L} = \frac{i\dot{\rho}}{2} - \rho\dot{\eta} - \frac{\rho}{2m}(\nabla\eta)^2 - \frac{(\nabla\rho)^2}{8m\rho} - \frac{g}{2}(\rho - \rho_0)^2. \quad (3.3)$$

If we drop out the first term as a total derivative, and define

$$\mathcal{L}_1 \equiv -\rho\dot{\eta} - \frac{\rho}{2m}(\nabla\eta)^2, \quad (3.4)$$

$$\mathcal{L}_2 \equiv -\frac{(\nabla\rho)^2}{8m\rho} - \frac{g}{2}(\rho - \rho_0)^2, \quad (3.5)$$

then the Lagrangian can be written as

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2. \quad (3.6)$$

Pay attention to that in Eq. (3.1) the field η takes values in $\mathbb{R}/2\pi\mathbb{Z}$, but for now we temporarily release this condition, so that it has values in \mathbb{R} . The constraint will be imposed later, which will give another piece to the theory.

We see that the term $-\rho\dot{\eta}$ in \mathcal{L}_1 is not written in a Lorentz-invariant way. We can complete ρ to a four-vector $f^\mu = (\rho, \vec{f})$, where ρ is its zeroth component, and \vec{f} is an auxiliary field. Then one can show that

$$\mathcal{L}_1 + \frac{m}{2\rho} \left(\vec{f} - \frac{\rho}{m} \nabla\eta \right)^2 = -f^\mu \partial_\mu \eta + \frac{m}{2\rho} \vec{f}^2. \quad (3.7)$$

If we define the path integral measure to be

$$\int \mathcal{D}\vec{f} \exp \left[i \int d^4x \frac{m}{2\rho} \vec{f}^2 \right] = 1, \quad (3.8)$$

then

$$\begin{aligned} e^{i \int d^4x \mathcal{L}_1} &= \int \mathcal{D}\vec{f} \exp \left[i \int d^4x \left(\mathcal{L}_1 + \frac{m}{2\rho} \left(\vec{f} - \frac{\rho}{m} \nabla\eta \right)^2 \right) \right] \\ &= \int \mathcal{D}\vec{f} \exp \left[i \int d^4x \left(-f^\mu \partial_\mu \eta + \frac{m}{2\rho} \vec{f}^2 \right) \right]. \end{aligned} \quad (3.9)$$

Integrating out η , we obtain

$$\partial_\mu f^\mu = 0, \quad (3.10)$$

which can be solved by

$$f^\mu = \frac{1}{6} \epsilon^{\mu\nu\lambda\sigma} H_{\nu\lambda\sigma} \quad (3.11)$$

with $H_3 = dB_2$. Using this expression of f^μ , we obtain

$$\frac{m}{2\rho} \vec{f}^2 = \frac{m}{4\rho} \sum_{i,j} H_{0ij}^2. \quad (3.12)$$

To rewrite \mathcal{L}_2 we first split B_2 into the background part and the fluctuation part:

$$B_2 = B_2^{(0)} + b_2, \quad (3.13)$$

and correspondingly,

$$H_{\nu\lambda\sigma} = H_{\nu\lambda\sigma}^{(0)} + h_{\nu\lambda\sigma}. \quad (3.14)$$

Since

$$\rho = f^0 = \frac{1}{6}\epsilon^{0\nu\lambda\sigma}H_{\nu\lambda\sigma} = H_{123}, \quad (3.15)$$

we can rewrite \mathcal{L}_2 as

$$\mathcal{L}_2 = -\frac{g}{12}h_{ijk}^2 - \frac{(\nabla h_{ijk})^2}{48m\rho}. \quad (3.16)$$

After some steps we obtain

$$\begin{aligned} & \int \mathcal{D}\rho \mathcal{D}\eta \exp \left[i \int d^4x (\mathcal{L}_1 + \mathcal{L}_2) \right] \\ &= \int \mathcal{D}B_2 \exp \left[i \int d^4x \left(-\frac{g}{12}\eta^{\mu\alpha}\eta^{\nu\beta}\eta^{\lambda\gamma}h_{\mu\nu\lambda}h_{\alpha\beta\gamma} - \frac{(\nabla h_{ijk})^2}{48m\rho_0} \right) \right]. \end{aligned} \quad (3.17)$$

For the low-energy regime of a BEC system that we are interested in, we can drop the term $\sim (\nabla h_{ijk})^2$, because it contributes to the dispersion relation only in the UV regime:

$$\omega^2 = c_s^2 k^2 + \frac{\sim k^4}{m^2}. \quad (3.18)$$

Now we return to the point that the theory should be invariant under $\eta \rightarrow \eta + 2\pi$, which we have not taken into account so far. The difference comes in Eq. (3.9), where we cannot simply integrate out η . Instead

$$-f^\mu \partial_\mu \eta = -f^\mu \partial_\mu \eta_{\text{vortex}} - f^\mu \partial_\mu \eta_{\text{smooth}}, \quad (3.19)$$

and we can only integrate out η_{smooth} , which induces the constraint

$$\partial_\mu f^\mu = 0. \quad (3.20)$$

For η_{vortex} there is

$$\begin{aligned} -f^\mu \partial_\mu \eta_{\text{vortex}} &= -\frac{1}{2}\epsilon^{\mu\nu\lambda\sigma} \partial_\nu B_{\lambda\sigma} \partial_\mu \eta_{\text{vortex}} \\ &= -\frac{1}{2}B_{\lambda\sigma} \epsilon^{\lambda\sigma\mu\nu} \partial_\mu \partial_\nu \eta_{\text{vortex}} \\ &= \pi \int d^2\sigma \epsilon^{ab} \partial_a X^\lambda \partial_b X^\sigma B_{\lambda\sigma} \delta^4(x^\mu - X^\mu(\tau, \sigma)), \end{aligned} \quad (3.21)$$

where we used the relation

$$\epsilon^{\lambda\sigma\mu\nu} \partial_\mu \partial_\nu \eta = -2\pi \int d^2\sigma \epsilon^{\alpha\beta} \partial_\alpha X^\lambda \partial_\beta X^\sigma \delta^4(x^\mu - X^\mu). \quad (3.22)$$

One can integrate this expression over spacetime, which can be formally written as

$$-\int d^4x f^\mu \partial_\mu \eta_{\text{vortex}} = \mu_1 \int B_2 \quad (3.23)$$

with $\mu_1 = 2\pi$. Therefore, in the spacetime without boundary the Gross-Pitaevskii theory (2.11) can be mapped into an effective string theory:

$$\int \mathcal{D}B_2 \exp \left[i \int d^4x \left(-\frac{g}{12} \eta^{\mu\alpha} \eta^{\nu\beta} \eta^{\lambda\gamma} h_{\mu\nu\lambda} h_{\alpha\beta\gamma} - \frac{(\nabla h_{ijk})^2}{48m\rho_0} \right) + i\mu_1 \int B_2 \right], \quad (3.24)$$

where the second term can be dropped in the IR regime, i.e.,

$$Z = \int DB_{\mu\nu} \exp \left[i\pi \sum_i \int_{\Sigma_i} d\sigma d\tau \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu} - \frac{ig}{2} \int d^4x h_3^2 \right], \quad (3.25)$$

where $h_3^2 = h_{\mu\nu\lambda} h^{\mu\nu\lambda}/6$. This theory now becomes effectively Lorentz invariant. Moreover, to study the dynamics of the system, one can also add a tension term $\sim g_{\mu\nu} \partial_a X^\mu \partial^a X^\nu$ like in string theory [6, 8].

3.2 Duality with Boundary

In this subsection, we consider the duality map in the presence of dark solitons, which can be viewed as boundaries of the system. The setup is following. We start with a (3+1)-dimensional spacetime without boundary, and we put two parallel dark solitons in the bulk of the (3+1)D spacetime, which are separated in the z -direction with a distance L . The dark soliton plane can be viewed as a (2+1)D spacetime without boundary. In the space between two dark soliton planes there can be closed vortex lines and open vortex lines, and the endpoints of the open vortex lines have to attach to one of the dark soliton planes.

Now let us discuss the duality map in the presence of dark solitons, which is also discussed in Ref. [21] using an alternative approach. First, we repeat the same steps as in the previous subsection. Let us start with the Gross-Pitaevskii Lagrangian (2.11):

$$\mathcal{L}_{GP} = i\phi^\dagger \partial_t \phi - \frac{1}{2m} (\nabla \phi^\dagger) (\nabla \phi) - \frac{g}{2} (|\phi|^2 - \rho_0)^2.$$

With the parametrization (3.1):

$$\phi = \sqrt{\rho} e^{i\eta},$$

the original Gross-Pitaevskii Lagrangian (2.11) becomes the expression in Eq. (3.3):

$$\mathcal{L} = \frac{i\dot{\rho}}{2} - \rho\dot{\eta} - \frac{\rho}{2m} (\nabla \eta)^2 - \frac{(\nabla \rho)^2}{8m\rho} - \frac{g}{2} (\rho - \rho_0)^2.$$

We can drop the first term as a total derivative and separate the phase-dependent part (3.4) and the phase-independent part (3.5):

$$\begin{aligned} \mathcal{L}_1 &\equiv -\rho\dot{\eta} - \frac{\rho}{2m} (\nabla \eta)^2, \\ \mathcal{L}_2 &\equiv -\frac{(\nabla \rho)^2}{8m\rho} - \frac{g}{2} (\rho - \rho_0)^2. \end{aligned}$$

In the presence of a boundary, the phase η consists of two parts, η_1 and η_2 , where $\eta_1(t, x, y, z)$ is a function defined on the (3+1)D spacetime, while $\eta_2(t, x, y) = \tilde{\eta}(t, x, y)$ is a function defined only on the (2+1)D boundary, which can be the dark soliton plane. Both

of them may contain singularities. We can perform the duality map on the (3+1)D and the (2+1)D spacetime separately, i.e.,

$$S = \int d^4x [\mathcal{L}_1^{4D} + \mathcal{L}_2^{4D}] + \ell \int d^3x [\mathcal{L}_1^{3D} + \mathcal{L}_2^{3D}] , \quad (3.26)$$

where ℓ is a constant length scale due to the dimensional reason, which will be discussed later in this subsection.

The separation of the (3+1)D part and the (2+1)D part of the action can also be obtained in the following way. Instead of Eq. (3.1) let us use another parametrization:

$$\phi = p e^{i\eta} , \quad (3.27)$$

where $p = \sqrt{\rho}$. For a dark soliton given by Eq. (2.30), the background part of the factor p only depends on z :

$$p_0 = \sqrt{n} \tanh \left(\frac{z}{\sqrt{2}\ell} \right) . \quad (3.28)$$

It is usually said that there is a π -jump in the phase when across a dark soliton plane. This is just due to the fact that p changes sign from one side of the soliton to the other side, and one can absorb the sign into the phase using $-1 = \exp(i\pi)$. Equivalently, we can keep the sign change of p and consider the phase without a π -jump. Using the new parametrization (3.27), we can rewrite the GP Lagrangian (2.11) as

$$\mathcal{L} = ip\dot{p} - p^2\dot{\eta} - \frac{p^2}{2m}(\nabla\eta)^2 - \frac{(\nabla p)^2}{2m} - \frac{g}{2}(p^2 - p_c^2)^2 , \quad (3.29)$$

which can also be obtained by replacing ρ with p^2 in Eq. (3.3), and $p_c = \sqrt{n}$ is a constant. Again, the first term is a total derivative that can be dropped out. In Appendix A, we analyze Eq. (3.29) term by term under the assumption that p consists of both the background and the fluctuations, i.e.,

$$p = p_0 + \tilde{p} , \quad (3.30)$$

where we take p_0 to be the profile given by Eq. (3.28), which depends only on z , while we assume that \tilde{p} does not depend on z . The physical reason is that the dark soliton is very heavy, so that its longitudinal position is fixed, and there are no fluctuations in the longitudinal direction. As shown in Appendix A, in the background of a dark soliton, the action can be expressed as

$$\int d^4x \mathcal{L} = \ell \int d^3x \left[-\tilde{\rho}\dot{\eta} - \frac{1}{2m}(\tilde{\rho} + C)(\tilde{\nabla}\eta)^2 - \frac{1}{8m\tilde{\rho}}(\tilde{\nabla}\tilde{\rho})^2 - \frac{g}{2}(\tilde{\rho} - \tilde{\rho}_0)^2 \right] , \quad (3.31)$$

which justifies the separation presented in Eq. (3.26).

Before we discuss the (2+1)D duality map of the new effective action obtained above, we see that the (3+1)D duality is exactly the same as the one discussed in the previous subsection. We would like to emphasize that when we separate η into η_{smooth} and η_{singular} , the smooth part η_{smooth} does not feel the vortex lines or the dark solitons, hence we can still do the partial integration, and the boundary integral vanishes at infinity.

Now let us focus on the (2+1)D duality. In the following, all the fields with tilde ($\tilde{}$) are defined in (2+1)D. The steps are similar as before, and the only difference is that the auxiliary field f^a now is a 3-vector with $a \in \{t, x, y\}$. Therefore,

$$-f^a \partial_a \tilde{\eta} = -f^a \partial_a \tilde{\eta}_{\text{smooth}} - f^a \partial_a \tilde{\eta}_{\text{singular}}. \quad (3.32)$$

The smooth part $\tilde{\eta}_{\text{smooth}}$ does not feel the vortices caused by the endpoints of vortex lines, and it is well-defined on the whole dark soliton plane. Integrating it out, we obtain

$$\partial_a f^a = 0, \quad (3.33)$$

which can be solved by

$$f^a = \frac{1}{2} \epsilon^{abc} F_{bc} \quad \text{with} \quad F_{ab} = \frac{1}{2} (\partial_a A_b - \partial_b A_a). \quad (3.34)$$

Consequently,

$$\begin{aligned} \exp \left[i \int d^3x \mathcal{L}_1^{3D} \right] &= \int \mathcal{D}f^a \exp \left\{ i\ell \int d^3x \left[-f^a \partial_a \tilde{\eta} + \frac{m}{2\tilde{\rho}} f^{\hat{a}} f_{\hat{a}} \right] \right\} \\ &= \int \mathcal{D}f^a \exp \left\{ i\ell \int d^3x \left[-f^a \partial_a \tilde{\eta}_{\text{singular}} + \frac{m}{2\tilde{\rho}} F_{0\hat{a}}^2 \right] \right\} \\ &= \int \mathcal{D}f^a \exp \left\{ i\ell \int d^3x \left[-f^a \partial_a \tilde{\eta}_{\text{singular}} + \frac{m}{2\tilde{\rho}} \tilde{F}_{0\hat{a}}^2 \right] \right\}, \end{aligned} \quad (3.35)$$

and

$$\mathcal{L}_2^{3D} = -\frac{(\nabla \tilde{F}_{\hat{a}\hat{b}})^2}{16m\tilde{\rho}} - \frac{g}{4} \tilde{F}_{\hat{a}\hat{b}}^2, \quad (3.36)$$

where $\hat{a}, \hat{b} \in \{x, y\}$. Together, they form

$$\begin{aligned} &\int \mathcal{D}\tilde{\rho} \mathcal{D}\tilde{\eta} \exp \left\{ i\ell \int d^3x \left[-\tilde{\rho} \dot{\tilde{\eta}} - \frac{\tilde{\rho}}{2m} (\nabla \tilde{\eta})^2 - \frac{(\nabla \tilde{\rho})^2}{8m\tilde{\rho}} - \frac{g}{2} (\tilde{\rho} - \rho_0)^2 \right] \right\} \\ &= \int \mathcal{D}A_a \exp \left\{ i\ell \int d^3x \left[-f^a \partial_a \tilde{\eta}_{\text{singular}} - \frac{g}{4} \tilde{\eta}^{ab} \tilde{\eta}^{cd} \tilde{F}_{ac} \tilde{F}_{bd} - \frac{(\nabla \tilde{F}_{\hat{a}\hat{b}})^2}{16m\tilde{\rho}} \right] \right\}. \end{aligned} \quad (3.37)$$

Now let us consider the term $-f^a \partial_a \tilde{\eta}_{\text{singular}}$. The singular part $\tilde{\eta}_{\text{singular}}$ is caused by the endpoints of the vortex lines in (3+1)D spacetime, which can be viewed as vortices in (2+1)D. Since we have the solution $f^a = \epsilon^{abc} \partial_b A_c$, we can plug it into the singular term, then we obtain

$$\begin{aligned} &i\ell \int d^3x [-f^a \partial_a \tilde{\eta}_{\text{singular}}] \\ &= -i\ell \int d^3x \epsilon^{abc} (\partial_b A_c) \partial_a \tilde{\eta}_{\text{singular}} \\ &= -i\ell \int d^3x \epsilon^{cab} A_c (\partial_a \partial_b \tilde{\eta}_{\text{singular}}) \\ &= -2\pi i\ell \int d^3x A_c j^c, \end{aligned} \quad (3.38)$$

where we have defined a vortex current:

$$j^c \equiv \frac{1}{2\pi} \epsilon^{cab} \partial_a \partial_b \tilde{\eta}_{\text{singular}} , \quad (3.39)$$

which satisfies

$$\int d^2x j^0 = \frac{1}{2\pi} \int d^2x \epsilon^{\hat{a}\hat{b}} \partial_{\hat{a}} \partial_{\hat{b}} \eta_{\text{singular}} = \frac{1}{2\pi} \int d^2x \nabla \times (\nabla \eta_{\text{singular}}) = \frac{1}{2\pi} \oint d\vec{x} \cdot \nabla \eta_{\text{singular}} = 1 . \quad (3.40)$$

Because the vortices on the dark soliton plane can also be viewed as the endpoints of the vortex lines in the (3+1)D spacetime, we may also use the following relation in (3+1)D:

$$\epsilon^{\lambda\sigma\mu\nu} \partial_\mu \partial_\nu \eta = -2\pi \int d^2\sigma \epsilon^{\alpha\beta} \partial_\alpha X^\lambda \partial_\beta X^\sigma \delta^4(x^\mu - X^\mu) \quad (3.41)$$

to obtain

$$\begin{aligned} & i\ell \int d^3x [-f^a \partial_a \tilde{\eta}_{\text{singular}}] \\ &= -i\ell \int d^3x \epsilon^{zcab} A_c (\partial_a \partial_b \tilde{\eta}_{\text{singular}}) \\ &= 2\pi i\ell \int d^3x \int d\tau d\sigma A_a \epsilon^{\alpha\beta} \partial_\alpha X^z \partial_\beta X^a \frac{1}{L} \delta^3(x - X) \\ &= 2\pi i\ell \int d\tau A_a \partial_\tau X^a , \end{aligned} \quad (3.42)$$

where the 3D δ -function is related to the 4D δ -function in the following way:

$$\delta^4(x - X) = \frac{1}{L} \delta^3(x - X) . \quad (3.43)$$

In the last step we used the fact that only $\frac{\partial X^z}{\partial \sigma} = \frac{\partial z}{\partial \sigma}$ is nonvanishing when the vortex line is perpendicular to the dark soliton, i.e. $z \parallel \sigma$, hence

$$\int d\sigma \partial_\sigma X^z = \int d\sigma \frac{\partial z}{\partial \sigma} = \int dz = L , \quad (3.44)$$

where L is the distance between two parallel dark soliton planes.

There are still two issues that we have to carefully address. One is the dimensionality. In the following we list the mass dimensions of various fields and parameters:

$$[\rho] = 3 , \quad [\eta] = 0 , \quad [m] = 1 , \quad [g] = -2 , \quad (3.45)$$

$$[H] = 3 , \quad [B] = 2 , \quad [\tilde{F}] = 3 , \quad [A] = 2 . \quad (3.46)$$

Conventionally, the gauge field A has mass dimension 1, which can be achieved by absorbing the length scale ℓ into A , i.e.,

$$\ell A_a \rightarrow A_a , \quad (3.47)$$

to make it of dimension 1. Also, conventionally the 2-form gauge field $B_{\mu\nu}$ is dimensionless. To achieve it, we can separate a dimensionful constant from it, i.e.,

$$B_{\mu\nu} \rightarrow \frac{1}{2\pi\alpha'} B_{\mu\nu} , \quad (3.48)$$

where $\alpha' = \ell_s^2$ with ℓ_s denoting the string length scale.

The other issue is that we did the (3+1)D duality and the (2+1)D duality separately. Although they seem to be independent of each other, the dualities are related through a boundary integral in (3+1)D. Let us recall for the (3+1)D case

$$\mathcal{L}^{4D} \supset -f^\mu \partial_\mu \eta_{\text{smooth}} - f^\mu \partial_\mu \eta_{\text{vortex}}, \quad (3.49)$$

and we partially integrated the first term and then integrated out η_{smooth} to obtain the equation $\partial_\mu f^\mu = 0$. During the derivation we dropped a boundary term

$$- \int d^4x \partial_\mu (f^\mu \eta_{\text{smooth}}), \quad (3.50)$$

which vanishes when the dark soliton is absent. In the presence of the dark soliton, this boundary term becomes

$$\begin{aligned} - \int d^4x \partial_\mu (f^\mu \eta_{\text{smooth}}) &= - \int d^3x f^\mu \tilde{\eta}_{\text{smooth}} \\ &= - \int d^3x \epsilon^{zabc} \frac{1}{2} (\partial_a B_{bc}) \tilde{\eta}_{\text{smooth}} \\ &= \int d^3x \epsilon^{abc} \frac{1}{2} B_{bc} \partial_a \tilde{\eta}_{\text{smooth}}, \end{aligned} \quad (3.51)$$

which should be combined with the term $-f^a \partial_a \tilde{\eta}_{\text{smooth}}$ in the (2+1)D duality. Hence, precisely speaking, in the (2+1)D duality the terms similar to the ones in Eq. (3.49) should be

$$\mathcal{L}^{3D} \supset - \left(f^a - \frac{1}{2} \epsilon^{abc} B_{bc} \right) \partial_a \tilde{\eta}_{\text{smooth}} - f^a \partial_a \tilde{\eta}_{\text{singular}}. \quad (3.52)$$

Partially integrating the first term, instead of $\partial_a f^a = 0$ we will obtain

$$\partial_a \left(f^a - \frac{1}{2} \epsilon^{abc} B_{bc} \right) = 0. \quad (3.53)$$

The solution to this equation is

$$f^a - \frac{1}{2} \epsilon^{abc} B_{bc} = \frac{1}{2} \epsilon^{abc} F_{bc} \quad \text{with} \quad F_{bc} = \frac{1}{2} (\partial_b A_c - \partial_c A_b). \quad (3.54)$$

Consequently,

$$f^a = \frac{1}{2} \epsilon^{abc} (F_{bc} + B_{bc}). \quad (3.55)$$

Hence, in the (2+1)D duality that we discussed above F should be replaced by $F+B$. After introducing some length scales to match the conventional dimensions, the combination should be

$$F_{ab} + \frac{1}{2\pi\alpha'} B_{ab}. \quad (3.56)$$

Therefore, the final expression of the (2+1)D dual theory is

$$\begin{aligned} & \int \mathcal{D}A_a \exp \left\{ -2\pi i \int d^3x A_a j^a + \frac{i}{\ell} \int d^3x \left[-\frac{g}{4} (\tilde{F} + \frac{1}{2\pi\alpha'} \tilde{B})^2 - \frac{(\nabla \tilde{F}_{\hat{a}\hat{b}} + \frac{1}{2\pi\alpha'} \nabla b_{\hat{a}\hat{b}})^2}{16m\tilde{\rho}} \right] \right\} \\ &= \int \mathcal{D}A_a \exp \left\{ 2\pi i \int d\tau A_a \partial_\tau X^a + \frac{i}{\ell} \int d^3x \left[-\frac{g}{4} (\tilde{F} + \frac{1}{2\pi\alpha'} \tilde{B})^2 - \frac{(\nabla \tilde{F}_{\hat{a}\hat{b}} + \frac{1}{2\pi\alpha'} \nabla b_{\hat{a}\hat{b}})^2}{16m\tilde{\rho}} \right] \right\}, \end{aligned} \quad (3.57)$$

where we can drop the last term in the IR regime, and

$$(\tilde{F} + \frac{1}{2\pi\alpha'}\tilde{B})^2 \equiv \tilde{\eta}^{ab}\tilde{\eta}^{cd} \left(\tilde{F}_{ac} + \frac{1}{2\pi\alpha'}\tilde{B}_{ac} \right) \left(\tilde{F}_{bd} + \frac{1}{2\pi\alpha'}\tilde{B}_{bd} \right), \quad (3.58)$$

where again \tilde{F} and \tilde{B} are fluctuations of F and B respectively. The full theory should be the combination of the action above with the one from the (3+1)D duality, which has the following expression in the IR regime:

$$Z = \int DB_{\mu\nu} DA_a \exp \left[\frac{i\eta}{2} \sum_i \int_{\Sigma_i} d\sigma d\tau \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu} - \frac{ig}{2} \int d^4x h_3^2 \right. \\ \left. + i\eta \sum_j \int_{\partial\Sigma_j} d\tau A_a \partial_\tau X^a - \frac{ig}{4\ell} \int d^3x (\tilde{F} + \tilde{B})^2 \right], \quad (3.59)$$

where $\eta = 2\pi\hbar$, $\alpha, \beta \in \{\tau, \sigma\}$, and Σ_i is the worldsheet spanned by the i -th vortex line with boundaries $\partial\Sigma_j$. The summation over $\partial\Sigma_j$ includes all the endpoints $X^a = X^a(\tau)$ of vortex lines attached to dark solitons. $H_{\mu\nu\lambda} \equiv \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu} = H_{\mu\nu\lambda}^0 + h_{\mu\nu\lambda}$, where $H_{\mu\nu\lambda}^0$ is the background field with $H_{123}^0 = \rho_0$, and the fluctuations are given by $h_3^2 = h_{\mu\nu\lambda} h^{\mu\nu\lambda}/6$ with the metric $\eta_{\mu\nu} = \text{diag}\{-c_s^2, 1, 1, 1\}$ and the speed of sound $c_s = \sqrt{g\rho_0/m}$. \tilde{F} and \tilde{B} are the fluctuations of F and B on the soliton plane respectively.

We would like to emphasize that the effective action (3.59) is invariant under the following gauge transformations (see e.g. Ref. [22]):

$$B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu, \\ A_a \rightarrow A_a - \Lambda_a, \quad (3.60)$$

where Λ_μ are transformation parameters.

3.3 Generalizations

In this section up to now, we have focused on the (3+1)D case, which can be generalized to other dimensions. Let us briefly discuss the generalization in the following.

We start with the Abelian Higgs model (2.1):

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}|(\partial_\mu + ieA_\mu)\phi|^2 - \lambda(|\phi|^2 - |\phi_0|^2)^2. \quad (3.61)$$

To simplify our discussion of the duality, we can assume $|\phi| \approx |\phi_0|$ everywhere so that $V(\phi\phi^\dagger) = -\lambda(|\phi|^2 - |\phi_0|^2)^2 \approx 0$. We therefore only focus on the relevant terms

$$\mathcal{L} \supset -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}|(\partial_\mu + ieA_\mu)\phi|^2, \quad (3.62)$$

and employ the parametrization (3.27):

$$\phi = p e^{i\eta}, \quad (3.63)$$

where $p \approx |\phi_0|$. Since in the approximation considered above

$$D_\mu \phi = \phi (i(\partial_\mu \eta + eA_\mu) + \partial_\mu \ln p) \approx i\phi (\partial_\mu \eta + eA_\mu), \quad (3.64)$$

the terms in Eq. (3.62) become

$$\mathcal{L} \supset -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}p^2(\partial_\mu\eta + eA_\mu)^2. \quad (3.65)$$

The field η takes values in $\mathbb{R}/2\pi\mathbb{Z}$ and has the gauge transformation $\eta \mapsto \eta - \lambda$ with $\lambda \in \mathbb{R}/2\pi n^{-1}\mathbb{Z}$. This is compensated by the gauge field transformation $A_\mu \mapsto A_\mu + \frac{1}{e}\partial_\mu\lambda$. Notice that in this formalism the presence of the vortex is encoded in the properties of gauge transformation, and therefore it is the gauge field A_μ which has the non-smooth behavior associated with the nontrivial topology. In the low-energy limit $|\phi_0| \rightarrow \infty$, the gauge symmetry is broken to \mathbb{Z}_n , so that $A_\mu = -\frac{1}{e}\partial_\mu\eta$ and $F_{\mu\nu} = 0$. The low-energy physics is described by the action

$$-\frac{1}{2}p^2(\partial_\mu\eta + eA_\mu)^2, \quad (3.66)$$

where $\eta = \eta_{\text{smooth}}$, or alternatively, after the substitution $A_\mu = -\frac{1}{e}\partial_\mu\eta_{\text{vortex}}$ described by

$$-\frac{1}{2}p^2(\partial_\mu\eta)^2, \quad (3.67)$$

where $\eta = \eta_{\text{smooth}} + \eta_{\text{vortex}}$. We can introduce an auxiliary field ζ^μ , so that the expression above can be rewritten as

$$-\frac{1}{2p^2}\zeta_\mu^2 + \zeta^\mu(\partial_\mu\eta), \quad (3.68)$$

or alternatively,

$$\mathcal{L} \supset -\frac{1}{2p^2}\zeta_\mu^2 + \zeta^\mu(\partial_\mu\eta_{\text{smooth}} + \partial_\mu\eta_{\text{vortex}}). \quad (3.69)$$

Integrating out η_{smooth} will impose

$$\partial_\mu\zeta^\mu = 0. \quad (3.70)$$

In the language of differential forms, the action (3.69) can be rewritten as

$$-\frac{1}{2p^2}\zeta \wedge *\zeta + \zeta \wedge *(d\eta_{\text{smooth}} + d\eta_{\text{vortex}}), \quad (3.71)$$

and the constraint (3.70) becomes

$$d*\zeta = 0, \quad (3.72)$$

which can be solved by $\zeta = *da$, where a is a $(D-2)$ -form, in order that ζ is a one-form. a and $f = da = *\zeta$ can be viewed as the gauge field and the field strength respectively, and the gauge transformation is

$$a \rightarrow a + d\Lambda, \quad (3.73)$$

where Λ is a $(D-3)$ -form. Finally, after the duality the relevant term (3.69) in the Lagrangian becomes

$$\mathcal{L} \supset -\frac{1}{2p^2}f \wedge *f + f \wedge *(d\eta_{\text{vortex}}), \quad (3.74)$$

which is just

$$-\frac{1}{2p^2}f_{\mu\dots}^2 + \epsilon^{\mu\nu\lambda\dots}\partial_\nu a_{\lambda\dots}(\partial_\mu\eta_{\text{vortex}}).$$

After a partial integration, it becomes

$$-\frac{1}{2p^2}f_{\mu\dots}^2 + a_{\mu\dots}j_{\text{vortex}}^{\mu\dots},$$

where $j_{\text{vortex}}^{\lambda\dots} = \epsilon^{\lambda\mu\nu\dots}\partial_\mu\partial_\nu\theta_{\text{vortex}}$ is a $(D-2)$ -form current, which is nonvanishing, because θ_{vortex} is not globally defined. We remark that in this section D is the dimension of spacetime in which the theory is defined, whereas the effective dimension relevant to the definition of the vortex is two.

As an example, the current of a point particle $X^\mu(\tau)$ (with the proper time τ) is

$$j^\mu = \int d\tau \partial_\tau X^\mu(\tau) \delta^{(D)}(x - X(\tau)).$$

Analogously, the current of a string $X^\mu(\tau, \sigma)$ (with the worldsheet coordinates (τ, σ)) is

$$j^{\mu\nu} = \int d\tau d\sigma \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \delta^{(D)}(x - X(\tau, \sigma)).$$

The generalization for a D_p -brane, which is a $(p+1)$ -dimensional object, looks like

$$j^{\mu\nu\dots} = \int d\xi^{p+1} \epsilon^{\alpha\beta\dots} \partial_\alpha X^\mu \partial_\beta X^\nu \dots \delta^{(D)}(x - X(\xi)),$$

where now the current $j^{\mu\nu\dots}$ is a $(p+1)$ -form. This means that, if we want to interpret the vortices as charged objects coupled to a 1-form, a 2-form and a $(D-2)$ -form gauge field in 3-, 4- and D -dimensions respectively, they should in turn be viewed as particles, strings and D_{D-3} -branes respectively. For $D=2$ this analysis suggests that the dual description may be some kind of matrix theory.

Therefore, in principle the $(3+1)$ D duality discussed in the section can be generalized to other dimensions. The key point here is that the correct interpretation of the vortex as a geometrical object with a definite dimension determines what kind of kinetic term we can introduce.

4 Solitons and D-branes

In Section 2, we have discussed the relation between the relativistic commutative tachyon field theory, which can be viewed as the relativistic version of the Gross-Pitaevskii theory. The non-relativistic Gross-Pitaevskii theory can be mapped into an effective string theory, as discussed in Section 3.

In this section, we would like to first discuss the noncommutative tachyon field theory and find the classical soliton solutions to this theory, which can be identified with D-branes in string theory [19], and then we would like to relate the solitons in Gross-Pitaevskii theory with the noncommutative solitons through a Seiberg-Witten map. Hence, the solitons in Gross-Pitaevskii theory, e.g. dark solitons, can indeed be viewed as D-branes in the effective string theory.

Since we are interested in the time-independent solutions, the Lorentz-violating term in the non-relativistic Gross-Pitaevskii theory will be irrelevant. Moreover, the noncommutative tachyon field theory corresponds to the large B -field limit of string theory [19],

and in this limit the kinetic term can also be neglect. Hence, in both cases one does not need to distinguish the relativistic and the nonrelativistic version of the theory, as long as one focuses on the time-independed aspect.

4.1 Noncommutative Tachyon Field Theory

Now let us turn to the noncommutative tachyon field theory, whose noncommutative soliton solutions are discussed in Ref. [18].

In order to find the Seiberg-Witten map for scalars, it is more convenient to first gauge the scalar theory (2.11) and study the resulting Abelian Higgs model, and then we may turn off the gauge field to obtain the Seiberg-Witten map for the pure scalar theory. The commutative Abelian Higgs model was discussed in Section 2, and it is given by the Lagrangian (2.1):

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}|(\partial_\mu + ieA_\mu)\phi|^2 - \lambda(|\phi|^2 - |\phi_0|^2)^2,$$

By making the fields noncommutative, one obtains the noncommutative Abelian Higgs model. The Seiberg-Witten map for the noncommutative Abelian Higgs model was studied in Ref. [17].

As discussed in Ref. [9], the Seiberg-Witten map for the gauge field should preserve the gauge transformation relation, i.e.,

$$\hat{A}(A) + \hat{\delta}_\lambda \hat{A}(A) = \hat{A}(A + \delta_\lambda A), \quad (4.1)$$

where the hat ($\hat{}$) denotes the noncommutative fields, and the gauge transformation for the ordinary Yang-Mills theory is

$$\delta_\lambda A_i = \partial_i \lambda + i[\lambda, A_i], \quad (4.2)$$

while for the noncommutative Yang-Mills theory:

$$\hat{\delta}_\lambda \hat{A}_i = \partial_i \hat{\lambda} + i\hat{\lambda} \star \hat{A}_i - i\hat{A}_i \star \hat{\lambda}. \quad (4.3)$$

Solving the constraint (4.1), one obtains

$$\begin{aligned} \hat{A}_i(A) &= A_i - \frac{1}{4}\theta^{kl}\{A_k, \partial_l A_i + F_{lk}\} + \mathcal{O}(\theta^2), \\ \hat{\lambda}(\lambda, A) &= \lambda + \frac{1}{4}\theta^{kl}\{\partial_k \lambda, A_l\} + \mathcal{O}(\theta^2). \end{aligned} \quad (4.4)$$

Similarly, for the scalar field we can also require that the gauge transformation relation should be preserved under the Seiberg-Witten map for both the commutative and the noncommutative fields, i.e.,

$$\hat{\phi}(\phi) + \hat{\delta}_\lambda \hat{\phi}(\phi) = \hat{\phi}(\phi + \delta_\lambda \phi, A + \delta_\lambda A). \quad (4.5)$$

Solving this equation for the Abelian gauge field at the order θ , one obtains [17]:

$$\hat{\phi} = \phi - \frac{1}{2}\theta^{kl}A_k\partial_l\phi + \mathcal{O}(\theta^2). \quad (4.6)$$

Now, we can turn off the gauge field to obtain the commutative and the noncommutative scalar field theory as follows:

$$\mathcal{L}_c = -\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \lambda(|\phi|^2 - |\phi_0|^2)^2, \quad (4.7)$$

$$\mathcal{L}_{\text{nc}} = -\frac{1}{2}(\partial_\mu \phi) \star (\partial^\mu \phi) - \lambda(\bar{\phi} \star \phi - \bar{\phi}_0 \star \phi_0)^2, \quad (4.8)$$

where the subscript “c” and “nc” stand for the commutative and the noncommutative theory respectively. The commutative one can be viewed as the relativistic version of the Gross-Pitaevskii theory, while the noncommutative one is just the noncommutative tachyon field theory that we are looking for.

The Seiberg Witten map for the scalar is trivial when turning off the gauge field:

$$\hat{\phi} = \phi. \quad (4.9)$$

However, the noncommutative scalar field still obeys the star product.

The noncommutative tachyon field theory has its origin in string theory. As explained in Ref. [9], we can turn on a constant B -field in the open string action, and the new open string action is equivalent to a noncommutative theory without the B -field. To relate the ordinary action with the noncommutative action, one only needs to perform the following replacements:

$$\begin{aligned} g_{\mu\nu} &\rightarrow G_{\mu\nu} = g_{\mu\nu} - (2\pi\alpha')^2 (Bg^{-1}B)_{\mu\nu}, \\ g_s &\rightarrow G_s = g_s \left(\frac{\det G}{\det (g + 2\pi\alpha' B)} \right)^{1/2}, \\ A(x) B(x) &\rightarrow A \star B = e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu\partial_\nu} A(x) B(x')|_{x=x'}, \end{aligned} \quad (4.10)$$

where

$$\theta^{\mu\nu} = -(2\pi\alpha')^2 \left(\frac{1}{g + 2\pi\alpha' B} B \frac{1}{g - 2\pi\alpha' B} \right)^{\mu\nu}. \quad (4.11)$$

To simplify the calculations, we can also consider the large noncommutativity limit, which is

$$\alpha' B_{ij} \rightarrow \infty, \quad g_{ij} \text{ fixed}. \quad (4.12)$$

The effective action for the tachyon field has the form

$$S = \frac{C}{g_s} \int d^D x \sqrt{g} \left[\frac{1}{2} f(T) g^{\mu\nu} \partial_\mu T \partial_\nu T - V(T) + \dots \right]. \quad (4.13)$$

When we turn on the B -field, the effective action becomes

$$S = \frac{C}{G_s} \int d^D x \sqrt{G} \left[\frac{1}{2} f(T) G^{\mu\nu} \partial_\mu T \partial_\nu T - V(T) + \dots \right], \quad (4.14)$$

where the \star -products of fields are implied.

For our discussion on the effective string theory obtained from the duality map, the string action is given by Eq. (3.25) with a boundary. Since we focus on the region around the dark soliton, the B -field can be approximately viewed as constant, which implies that the field strength term $\sim h_3^2$ in Eq. (3.25) can be neglected. Hence, the effective action (3.25) is exactly the same as the standard string action in the large B -field limit, and the analysis above holds in our case.

4.2 Identification of Classical Solutions

It was discussed in Ref. [19] that noncommutative solitons in tachyon field theory can be identified with D-branes in string theory. Let us briefly review the identification in the following.

We consider the equation for the static solitons

$$\frac{dV}{dT} = 0, \quad (4.15)$$

where $V(T)$ is the tachyon potential discussed in the end of the previous subsection. The following analysis is valid for a very broad class of tachyon potentials, in particular, the potential in the Gross-Pitaevskii theory. For simplicity, we can shift and rescale the tachyon field T , such that the potential $V(T)$ has a local maximum $V = 1$ at T_* and a local minimum 0 at $T = 0$.

The soliton solutions were constructed in the large noncommutativity limit in Ref. [18]. To construct them, one needs a field ϕ_0 satisfying

$$\phi_0 \star \phi_0 = \phi_0, \quad (4.16)$$

then any function F with the form $F(x) = \sum_{n=1}^{\infty} a_n x^n$ has the property

$$F(\lambda \phi_0) = F(\lambda) \phi_0. \quad (4.17)$$

We will assume that the tachyon potential $V(T)$ has this form, hence

$$\left. \frac{dV}{dT} \right|_{t=\lambda \phi} = \left(\left. \frac{dV}{dT} \right|_{t=\lambda} \right) \phi_0. \quad (4.18)$$

The simplest function satisfying the condition (4.16) was found in Ref. [18]:

$$\phi_0(r) = 2 e^{-r^2/\theta}, \quad r^2 = x_1^2 + x_2^2, \quad (4.19)$$

where $\theta = 1/B$ and $B = B_{12}$. It can be seen as follows. We start with a Gaussian packet

$$\psi_{\Delta}(r) = \frac{1}{\pi \Delta^2} e^{-\frac{r^2}{\Delta^2}}, \quad r^2 = x^2 + y^2. \quad (4.20)$$

The star product can be easily computed in the momentum space:

$$\begin{aligned} \left(\tilde{\psi}_{\Delta} \star \tilde{\psi}_{\Delta} \right) (p) &= \frac{1}{(2\pi)^2} \int d^2 k \tilde{\psi}_{\Delta}(k) \tilde{\psi}_{\Delta}(p-k) e^{\frac{i}{2} \epsilon_{\mu\nu} k^{\mu} (p-k)^{\nu}} \\ &= \frac{1}{2\pi \Delta^2} e^{-\frac{p^2}{8} \left(\Delta^2 + \frac{1}{\Delta^2} \right)}, \end{aligned} \quad (4.21)$$

where

$$\tilde{\psi}_{\Delta}(k) \equiv \int d^2 x e^{ik \cdot x} \psi_{\Delta}(x) = e^{-\frac{k^2 \Delta^2}{4}}. \quad (4.22)$$

Therefore,

$$(\psi_{\Delta} \star \psi_{\Delta})(x) = \frac{1}{\pi^2 \Delta^2 \left(\Delta^2 + \frac{1}{\Delta^2} \right)} \exp \left[\frac{-2r^2}{\Delta^2 + \frac{1}{\Delta^2}} \right]. \quad (4.23)$$

If we define

$$\phi_0(x) \equiv 2\pi\psi_1(x) = 2e^{-r^2}, \quad (4.24)$$

then

$$(\phi_0 \star \phi_0)(x) = \phi_0(x). \quad (4.25)$$

Hence, the tachyon potential should satisfy

$$V(T) = V(T) \phi_0(r), \quad (4.26)$$

and for the potential that we discussed before the soliton solution is just $V(T_*) \phi_0(r)$. This construction can be generalized to arbitrary even codimension solitons by replacing $r^2 = x_1^2 + x_2^2$ with $r^2 = x_1^2 + x_2^2 + \dots + x_{2q}^2$, and we would like to interpret this soliton as a $D_{(D-2q)}$ -brane.

Based on our discussions in the previous subsection, especially the relation between the commutative and the noncommutative tachyon field theories, we would like to identify the noncommutative solitons and the D -branes also with commutative solitons, e.g. dark solitons in Gross-Pitaevskii theory.

4.2.1 D-brane Tension

To verify the identification of the noncommutative solitons in tachyon field theory with the D_p -branes in string theory, various checks were done in Ref. [19]. For instance, one can compute the D-brane tension from the tachyon effective action, and compare it with the one obtained in string theory. Let us briefly review this comparison in the following.

First, in the large noncommutativity limit one can neglect the derivative term in the tachyon effective action, then it becomes

$$S = -\frac{C}{G_s} \int d^D x \sqrt{G} V(T). \quad (4.27)$$

The tachyon potential at the soliton solution $T = T_*$ has the value $V(T_*) \phi_0(r)$, therefore,

$$S = -\frac{C V(T_*)}{G_s} \int d^{D-2} x \int d^2 x \sqrt{G} \phi_0(r) = -\frac{2\pi\theta C V(T_*)}{G_s} \int d^{D-2} x \sqrt{G}. \quad (4.28)$$

From Eq. (4.10) we obtain for the large B -field the relation between the open string coupling G_s and the closed string coupling g_s :

$$G_s = \frac{g_s \sqrt{G}}{2\pi\alpha' B \sqrt{g}}. \quad (4.29)$$

For the tachyon potential we consider $V(T_*) = 1$. Taking into account $\theta = 1/B$, we obtain

$$S = -(2\pi)^2 \alpha' \frac{C}{g_s} \int d^{D-2} x \sqrt{g}. \quad (4.30)$$

Hence, the soliton tension is

$$T_{\text{sol}} = (2\pi)^2 \alpha' \frac{C}{g_s} = (2\pi)^2 \alpha' T_D = T_{D-2}, \quad (4.31)$$

where $C = T_D g_s$. This result is consistent with the one for the bosonic D_p-branes:

$$T_p = (2\pi\sqrt{\alpha'})^{D-p} T_D. \quad (4.32)$$

From the Gross-Pitaevskii theory point of view, the dark soliton is a domain wall solution, whose energy is expressed as a one-dimensional integral in Eq. (2.22). As discussed in Ref. [4], to prevent the long-wavelength instabilities we have to restrict the sizes of the transverse dimensions to be $\sim 2\pi\sqrt{\alpha'}$, where $\ell_s = \sqrt{\alpha'}$ is the characteristic length of the effective strings, i.e. the length of the vortex lines. Hence, the tension of the dark soliton also obeys the relation (4.32):

$$T_p = (2\pi\sqrt{\alpha'})^{D-p} T_D,$$

which supports the identification of dark solitons in Gross-Pitaevskii theory and D -branes in the effective string theory.

4.2.2 D-brane Interaction

As another check of the identification, in this subsection we compute the interaction between two parallel D-branes in the effective string theory. A similar computation in the ordinary string theory was done in Ref. [23], and summarized in e.g. Ref. [24].

The calculation is essentially to evaluate the amplitude of exchanging a closed string between two parallel D-branes (see Fig. 3), or equivalently to evaluate a 1-loop amplitude of open strings. For the picture of closed strings in the NS-NS sector, only the graviton and the dilaton were taken into account, because the antisymmetric B -field contributes at higher order. If one considers the type-II superstring, the contribution from the R-R sector will exactly cancel the one from the NS-NS sector, as discussed in Ref. [23]. However, the effective string theory discussed in this paper can be thought of as the large B -field limit of the ordinary string theory. Hence, the contributions from the graviton and the dilaton can be neglected, and only the B -field contributes to the potential between two parallel D-branes.

To compute the amplitude of exchanging the B -field between two parallel D-branes, we need the propagator of the B -field in the bulk and the coupling between the B -field and the D-brane. Following Ref. [24], we can read off the bulk propagator of the B -field from the effective action in the Einstein frame:

$$S^E = \frac{1}{2\kappa^2} \int d^D x (-\tilde{G})^{1/2} \left[\tilde{R} - \frac{4}{D-2} \nabla_\mu \tilde{\Phi} \nabla^\mu \tilde{\Phi} - \frac{1}{12} e^{-8\tilde{\Phi}/(D-2)} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{2(D-36)}{3\alpha'} e^{4\tilde{\Phi}/(D-2)} + \mathcal{O}(\alpha') \right], \quad (4.33)$$

where

$$\begin{aligned} \tilde{\Phi} &= \Phi - \Phi_0, \\ \tilde{G}_{\mu\nu} &= e^{\frac{4(\Phi_0 - \Phi)}{D-2}} G_{\mu\nu}, \end{aligned} \quad (4.34)$$

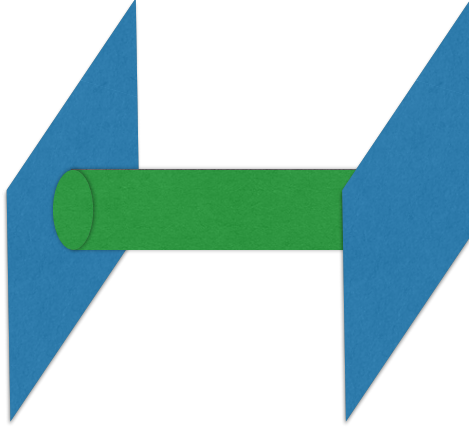


Figure 3. The exchange of a closed string between two parallel D-branes

and \tilde{R} is the corresponding Ricci scalar after the transformation. The terms relevant to the bulk propagator of the B -field are

$$S^E \supset -\frac{1}{24\kappa^2} \int d^D x B_{\mu\nu} \square B_{\mu\nu}, \quad (4.35)$$

and the bulk propagator of the B -field in momentum space is [25]:

$$\langle B_{\mu\nu} B_{\rho\sigma} \rangle = -\frac{6i\kappa^2}{k^2} (\delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\nu\rho} \delta_{\mu\sigma}). \quad (4.36)$$

To obtain the coupling between the B -field and the D-brane, we expand the DBI-action in the Einstein frame:

$$S_p^E = -\tau_p \int d^{p+1} \xi e^{-\tilde{\Phi}} \det \sqrt{e^{\frac{4\tilde{\Phi}}{D-2}} \tilde{G}_{ab} + B_{ab} + 2\phi\alpha' F_{ab}}, \quad (4.37)$$

where the indices a, b run over the $(p+1)$ -dimensions on the D-brane. The terms relevant to the coupling between the B -field and the D-brane are

$$S_p^E \supset -\frac{\tau_p}{4} \int d^{p+1} \xi B_{ab} B^{ab}. \quad (4.38)$$

From this coupling we see that the leading order contribution is already at 1-loop order, thus from field theory point of view we need to evaluate the following 1-loop graph:

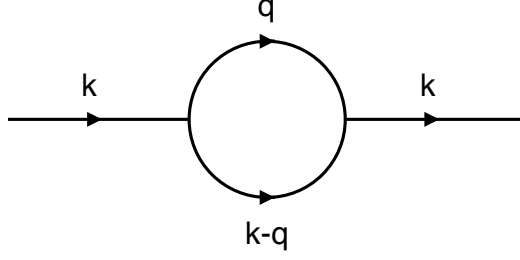


Figure 4. The 1-loop Feynman diagram of the B -field coupled to D-branes

The amplitude is

$$\begin{aligned}
i\mathcal{M}(k) &= \frac{1}{2} \left(-\frac{i\tau_p}{2} \right)^2 \int \frac{d^{D-p-1}q}{(2\pi)^{D-p-1}} \left(-\frac{6i\kappa^2}{q^2} \right) \left(-\frac{6i\kappa^2}{(k-q)^2} \right) \\
&\quad \cdot \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} \frac{\partial X^\rho}{\partial \xi^c} \frac{\partial X^\sigma}{\partial \xi^d} (\delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\nu\rho} \delta_{\mu\sigma}) \\
&\quad \cdot \frac{\partial X^{\bar{\mu}}}{\partial \xi^a} \frac{\partial X^{\bar{\nu}}}{\partial \xi^b} \frac{\partial X^{\bar{\rho}}}{\partial \xi^c} \frac{\partial X^{\bar{\sigma}}}{\partial \xi^d} (\delta_{\bar{\mu}\bar{\rho}} \delta_{\bar{\nu}\bar{\sigma}} - \delta_{\bar{\nu}\bar{\rho}} \delta_{\bar{\mu}\bar{\sigma}}) \\
&= \frac{9}{2} \tau_p^2 \kappa^4 [(p+1)^2 - 2(p+1)] \int \frac{d^{D-p-1}q}{(2\pi)^{D-p-1}} \frac{1}{q^2 (k-q)^2} \\
&= \frac{9}{2} \tau_p^2 \kappa^4 (p^2 - 1) \frac{i^{D-p-4}}{(4\pi)^{\frac{D-p-1}{2}}} \frac{\Gamma\left(\frac{5-D+p}{2}\right) \Gamma\left(\frac{D-p-3}{2}\right)^2}{\Gamma(D-p-3)} \frac{1}{k^{5-D+p}}. \tag{4.39}
\end{aligned}$$

To obtain the potential in the spacetime, we should apply the Born approximation and Fourier transform the amplitude $-\mathcal{M}(k)$.

In order to compare with the interactions between dark solitons, we consider the case $D = 4$, $p = 0$. As we discussed before, in order to make the dark soliton relatively stable and compatible with the Derrick's theorem, one has to restrict the size of transverse dimensions on the D-brane, i.e. to confine the system in a cylindrical geometry. Hence, in this computation the D-branes in real BEC systems can be thought of as D0-branes. After the Fourier transform of the amplitude, we obtain the potential between two parallel D-branes ($D = 4$, $p = 0$):

$$\begin{aligned}
V(x) &= i \frac{9}{16} \tau_p^2 \kappa^4 \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik \cdot x}}{k} \\
&= \frac{9 \tau_p^2 \kappa^4}{32\pi} \frac{\delta(x)}{x}. \tag{4.40}
\end{aligned}$$

We see that, strictly speaking the contribution of the B -field to the interaction between two D0-branes is given by a Dirac δ -function, i.e. a contact interaction. However, in reality the size of transverse dimensions on the D-brane is not zero, although negligible compared to the distance between two D-branes. Hence, the Dirac δ -function in Eq. (4.40) can be

understood as

$$\lim_{\ell \rightarrow 0} \frac{e^{-x^2/\ell^2}}{\sqrt{\pi\ell}}, \quad (4.41)$$

where ℓ is proportional to the size of transverse dimensions on the D-brane. Therefore, we expect the potential between two parallel dark solitons in BEC systems is a short-ranged repulsion and exponentially decaying. As far as we know, there is no analytical expression of the potential between two parallel dark solitons available in the literature, and the numerical results [26] are consistent with our results from string theory computation at qualitative level. More interestingly, some recent studies [27] in optical systems confirmed experimentally that, dark solitons can have attractions only when some nonlocal response is turned on, which is also consistent with our expectation from string theory, i.e., in the presence of a string tension term in the action, the exchange of the graviton and the dilaton will induce an attractive interaction between two parallel D-branes.

5 Discussions

In this paper we have discussed the duality map between the Gross-Pitaevskii theory and the (3+1)D effective string theory. We generalize the previous works [6, 8] to the spacetime with boundaries, which is also discussed in Ref. [21]. As a consequence, we identified the soliton solutions in the Gross-Pitaevskii theory and the D-branes in the effective string theory, and various checks have been made to test this identification. With this new perspective, one has an opportunity to test many results and predictions of string theory in real experiments and on the other hand bring in new ideas to the study of quantum fluids and cold atom systems.

We would like to explore more aspects of this duality and its relation to a real cold atom system at quantitative level. For instance, Ref. [4] started discussing the stability of the configuration of open vortex lines attached to the dark solitons, and we believe that a more detailed analysis of this dual picture can help us study the time evolution of D-brane decay. More interestingly, by introducing some fermionic fields an emergent supersymmetry can be realized in the cold atom systems. We hope that this can help stabilize the dark solitons, like the stable D-branes from the superstring theory.

From more theoretical point of view, the boson/vortex duality discussed in this paper is also of great interest. As we discussed in Subsection 3.3, the duality can be generalized to other dimensions. Since the (1+1)D Gross-Pitaevskii equation, also called the nonlinear Schrödinger equation, is an integrable model, we expect the integrability should be maintained in the dual theory. Also, it was known that the (1+1)D nonlinear Schrödinger equation is dual to a 2D topological Yang-Mills-Higgs model at quantum level [28]. By constructing the gravity dual of the 2D topological Yang-Mills-Higgs model, we expect that the D-branes in the supergravity theory correspond to the soliton solutions to the (1+1)D nonlinear Schrödinger equation. These results will be presented elsewhere [29].

Moreover, the identification of solitons and D-branes discussed in this paper can also be understood from the viewpoint of K-theory. As discussed in Ref. [24], in the annihilation of a D_p -brane and a anti- D_p -brane, if the tachyon field is given by a topologically stable

kink depending only on one of the dimensions inside the brane, then a $D_{(p-1)}$ -brane will be left over after the annihilation. In our case, the (anti-) D_p -brane can be viewed as the space-filling (anti-) D_3 -brane, while in the end we should see a D_2 -brane left, which can be identified as the dark soliton in the Gross-Pitaevskii theory. More details of this perspective and its applications to topological phases will be explored in the future work.

In stead of the boson/vortex duality, some recent works [30, 31] discussed the more general particle/vortex duality web, especially the dual of the fermionic field theory in (2+1)D. To apply these ideas to the (3+1)D Abelian Higgs model and understand the corresponding web of dualities will help us understand of the vacuum structure and the renormalization group flow of the theory, which we would like to pursue soon.

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A Some Details in the Duality Map

In this appendix, we present some details of the duality map in the presence of boundaries. Let us consider Eq. (3.29) term by term:

$$\mathcal{L} = ip\dot{p} - p^2\dot{\eta} - \frac{p^2}{2m}(\nabla\eta)^2 - \frac{(\nabla p)^2}{2m} - \frac{g}{2}(p^2 - p_0^2)^2,$$

where the first term is a total derivative that can be dropped.

- $-p^2\dot{\eta}$:

As we analyzed in Subsection 3.2, near the soliton plane p changes sign from one side to the other side, while η does not have a π -jump, or in other words, we remove the π -jump of the phase and allow p to change sign, and now the phase η behaves smoothly. Hence, when we consider the limit that the healing length goes to zero, the smooth functions such as $\dot{\eta}$ and $(\nabla\eta)^2$ will take their values at $z = z_0$, where z_0 is the longitudinal position of the dark soliton, i.e., the functions of the phase become z -independent in this limit. Therefore,

$$\begin{aligned} - \int d^4x (p_0 + \tilde{p})^2 \dot{\eta} &= - \left(\int_{z_0-\ell/2}^{z_0+\ell/2} dz p_0^2 \right) \left(\int d^3x \dot{\eta} \right) - \left(\int_{z_0-\ell/2}^{z_0+\ell/2} dz \right) \left(\int d^3x \tilde{p}^2 \dot{\eta} \right) \\ &\quad - 2 \left(\int_{z_0-\ell/2}^{z_0+\ell/2} dz p_0 \right) \left(\int d^3x \tilde{p} \dot{\eta} \right), \end{aligned} \tag{A.1}$$

where the last term vanishes due to $p_0(-x) = -p_0(x)$, and for the first term

$$\begin{aligned}
\int_{z_0-\ell/2}^{z_0+\ell/2} dz p_0^2 &= n \int_{z_0-\ell/2}^{z_0+\ell/2} dz \left[\tanh \left(\frac{z - z_0}{\sqrt{2}\ell} \right) \right]^2 \\
&= n \int_{-\ell/2}^{\ell/2} dz \left[\tanh \left(\frac{z}{\sqrt{2}\ell} \right) \right]^2 \\
&= n\ell \int_{-1/2}^{1/2} d\tilde{z} \left[\tanh \left(\frac{\tilde{z}}{\sqrt{2}} \right) \right]^2 \\
&= n\ell \left(1 - 2\sqrt{2} \tanh(1/2\sqrt{2}) \right) \\
&\equiv n\tilde{\ell}.
\end{aligned} \tag{A.2}$$

We defined $\tilde{z} \equiv z/\ell$, and in the last step we defined another length scale of the order of the healing length. Therefore,

$$- \int d^4x p_0^2 \dot{\eta} = -n\tilde{\ell} \int d^3x \dot{\eta}, \tag{A.3}$$

which is a total derivative, hence also can be dropped. What is remaining is

$$- \int d^4x (p_0 + \tilde{p})^2 \dot{\eta} = -\ell \int d^3x \tilde{p}^2 \dot{\eta}. \tag{A.4}$$

- $-\frac{p^2}{2m}(\nabla\eta)^2$:

Like in the previous case, the smooth function $(\nabla\eta)^2$ becomes z -independent in the small region around the dark soliton plane. Hence,

$$\begin{aligned}
- \int d^4x \frac{(p_0 + \tilde{p})^2}{2m} (\nabla\eta)^2 &= -\frac{1}{2m} \left(\int_{z_0-\ell/2}^{z_0+\ell/2} dz p_0^2 \right) \left(\int d^3x (\tilde{\nabla}\eta)^2 \right) \\
&\quad - \frac{1}{2m} \left(\int_{z_0-\ell/2}^{z_0+\ell/2} dz \right) \left(\int d^3x \tilde{p}^2 (\tilde{\nabla}\eta)^2 \right) \\
&\quad - \frac{1}{m} \left(\int_{z_0-\ell/2}^{z_0+\ell/2} dz p_0 \right) \left(\int d^3x \tilde{p} (\tilde{\nabla}\eta)^2 \right),
\end{aligned} \tag{A.5}$$

where $\tilde{\nabla}$ is the gradient operator on the coordinates (t, x, y) . Similar to the previous case, we obtain

$$\begin{aligned}
- \int d^4x \frac{(p_0 + \tilde{p})^2}{2m} (\nabla\eta)^2 &= -\frac{n\tilde{\ell}}{2m} \int d^3x (\tilde{\nabla}\eta)^2 - \frac{\ell}{2m} \int d^3x \tilde{p}^2 (\tilde{\nabla}\eta)^2 \\
&= -\frac{\ell}{2m} \int d^3x \left(\tilde{p}^2 + \frac{n\tilde{\ell}}{\ell} \right) (\tilde{\nabla}\eta)^2.
\end{aligned} \tag{A.6}$$

- $-\frac{(\nabla p)^2}{2m}$:

$$\begin{aligned}
-\int d^4x \frac{(\nabla p)^2}{2m} &= -\frac{1}{2m} \int_{z_0-\ell/2}^{z_0+\ell/2} dz \int d^3x \left[\left(\frac{\partial \tilde{p}}{\partial t} \right)^2 + \left(\frac{\partial \tilde{p}}{\partial x} \right)^2 + \left(\frac{\partial \tilde{p}}{\partial y} \right)^2 + \left(\frac{\partial p_0}{\partial z} \right)^2 \right] \\
&= -\frac{1}{2m} \left(\int_{z_0-\ell/2}^{z_0+\ell/2} dz \right) \int d^3x \left[\left(\frac{\partial \tilde{p}}{\partial t} \right)^2 + \left(\frac{\partial \tilde{p}}{\partial x} \right)^2 + \left(\frac{\partial \tilde{p}}{\partial y} \right)^2 \right] \\
&\quad - \frac{1}{2m} \left[\int_{z_0-\ell/2}^{z_0+\ell/2} dz \left(\frac{\partial p_0}{\partial z} \right)^2 \right] \left(\int d^3x \right), \tag{A.7}
\end{aligned}$$

where the second line contributes a constant, which can be dropped from the action, and the first line gives

$$-\frac{\ell}{2m} \int d^3x \left(\tilde{\nabla} \tilde{p} \right)^2. \tag{A.8}$$

- $-\frac{g}{2}(p^2 - p_c^2)^2$:

$$(p^2 - p_c^2)^2 = ((p_0 + \tilde{p})^2 - p_c^2)^2 = p_0^4 + 4p_0^3\tilde{p} + 6p_0^2\tilde{p}^2 + 4p_0\tilde{p}^3 + \tilde{p}^4 - 2p_0^2p_c^2 - 4p_0\tilde{p}p_c^2 - 2\tilde{p}^2p_c^2 + p_c^4. \tag{A.9}$$

After neglecting the terms that have odd powers in p_0 , we obtain the relevant terms

$$p_0^4 + 6p_0^2\tilde{p}^2 + \tilde{p}^4 - 2p_0^2p_c^2 - 2\tilde{p}^2p_c^2 + p_c^4, \tag{A.10}$$

where the terms independent of \tilde{p} contribute only constants after the integration over spacetime, which can be dropped from the action. The remaining terms are

$$\tilde{p}^4 - 2\tilde{p}^2(p_c^2 - 3p_0^2). \tag{A.11}$$

Hence,

$$\begin{aligned}
-\frac{g}{2} \int d^4x (p^2 - p_0^2)^2 &= -\frac{g}{2} \left(\int_{z_0-\ell/2}^{z_0+\ell/2} dz \right) \left(\int d^3x \tilde{p}^4 \right) + g \left(\int_{z_0-\ell/2}^{z_0+\ell/2} dz (p_c^2 - 3p_0^2) \right) \left(\int d^3x \tilde{p}^2 \right) \\
&= -\frac{g\ell}{2} \int d^3x \tilde{p}^4 + g(p_c^2\ell - 3n\tilde{\ell}) \int d^3x \tilde{p}^2 \\
&= -\frac{g\ell}{2} \int d^3x \left(\tilde{p}^2 - n \left(1 - \frac{3\tilde{\ell}}{\ell} \right) \right)^2 + \frac{g\ell}{2} \int d^3x \left(n - \frac{3n\tilde{\ell}}{\ell} \right)^2, \tag{A.12}
\end{aligned}$$

where we used $p_c = \sqrt{n}$, and the second term above is a constant, that can be dropped from the action. What is remaining after the integration is

$$-\frac{g\ell}{2} \int d^3x \left(\tilde{p}^2 - n \left(1 - \frac{3\tilde{\ell}}{\ell} \right) \right)^2, \tag{A.13}$$

where $\tilde{\ell} \equiv \ell (1 - 2\sqrt{2} \tanh(1/2\sqrt{2}))$, hence $(1 - 3\tilde{\ell}/\ell)$ is a positive constant. We can define

$$\tilde{p}_c \equiv \sqrt{n \left(1 - \frac{3\tilde{\ell}}{\ell}\right)}. \quad (\text{A.14})$$

Combining all the terms together, we obtain the action around a soliton plane

$$\begin{aligned} \int d^4x \mathcal{L} &= -\ell \int d^3x \tilde{p}^2 \dot{\eta} - \frac{\ell}{2m} \int d^3x \left(\tilde{p}^2 + \frac{n\tilde{\ell}}{\ell} \right) (\tilde{\nabla}\eta)^2 - \frac{\ell}{2m} \int d^3x (\tilde{\nabla}\tilde{p})^2 - \frac{g\ell}{2} \int d^3x (\tilde{p}^2 - \tilde{p}_c^2)^2 \\ &= \ell \int d^3x \left[-\tilde{p}^2 \dot{\eta} - \frac{1}{2m} (\tilde{p}^2 + C) (\tilde{\nabla}\eta)^2 - \frac{1}{2m} (\tilde{\nabla}\tilde{p})^2 - \frac{g}{2} (\tilde{p}^2 - \tilde{p}_c^2)^2 \right] \\ &= \ell \int d^3x \left[-\tilde{\rho} \dot{\eta} - \frac{1}{2m} (\tilde{\rho} + C) (\tilde{\nabla}\eta)^2 - \frac{1}{8m\tilde{\rho}} (\tilde{\nabla}\tilde{\rho})^2 - \frac{g}{2} (\tilde{\rho} - \tilde{\rho}_0)^2 \right], \end{aligned} \quad (\text{A.15})$$

where $C \equiv n\tilde{\ell}/\ell$ is a constant. In the last line we rewrite the theory in the variable $\tilde{\rho} = \sqrt{\tilde{p}}$. This action is very similar to the 3D part in the action (3.26) by restricting the Lagrangian (2.11) on a 3D space. The only difference is an additional term $-(C/2m)(\tilde{\nabla}\eta)^2$, but it does not affect the (2+1)D duality. The reason is following. In the duality map, we will introduce an auxiliary field f^a , and for $\tilde{\rho}' \equiv \tilde{\rho} + C$:

$$-\tilde{\rho} \dot{\eta} - \frac{\tilde{\rho}'}{2m} (\tilde{\nabla}\eta)^2 + \frac{m}{2\tilde{\rho}'} \left(f_a - \frac{\tilde{\rho}'}{m} \partial_a \eta \right)^2 = -\tilde{\rho} \dot{\eta} + \frac{m}{2\tilde{\rho}'} f^a f_a - f^a \partial_a \eta, \quad (\text{A.16})$$

where in the path integral

$$\int \mathcal{D}f^a \exp \left(i\ell \int d^3x \frac{m}{2\tilde{\rho}'} f^a f_a \right) = 1. \quad (\text{A.17})$$

Hence, $\tilde{\rho}'$ or consequently the constant C does not show up in the action after the duality map.

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